Alternating Path Algorithm with Party Hats

Júlia Kornai
julia.kornai@gmail.com

Dávid Szeszlér
szeszler@cs.bme.hu

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The Ósükösd Camps of the ELTE Radnóti Miklós High School

- talented and open-minded, but mostly not "mathy" children (Since then: chemist, doctor, economist, philosopher, poet, psychologist, political scientist,. . . — and: mathematician)

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  - show the experience and joy of discovery
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- altogether 2 or 3 times 90-120 minutes of math

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Playing the game:

- First immediately, right after announcing the rules.
- Before the second round groups are given time to discuss strategies.

1. The Plush Animal Game — Aims and Observations

Common discussion of strategies. Tipical:

- If I haven't got a plush, but I see one on the table that I like then I grab it.
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- Optional: exact terminology (bipartite graph, perfect matching)

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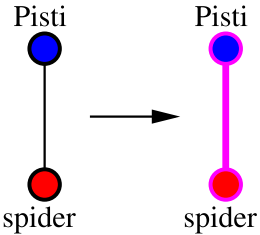

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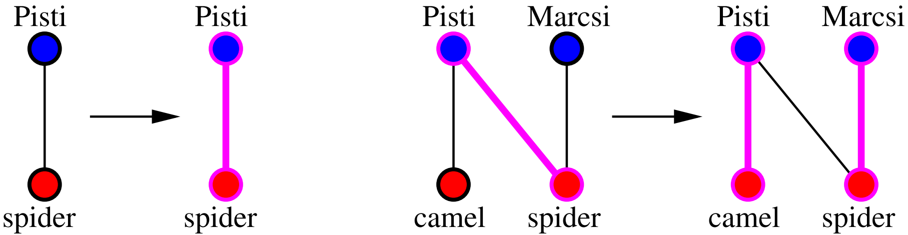

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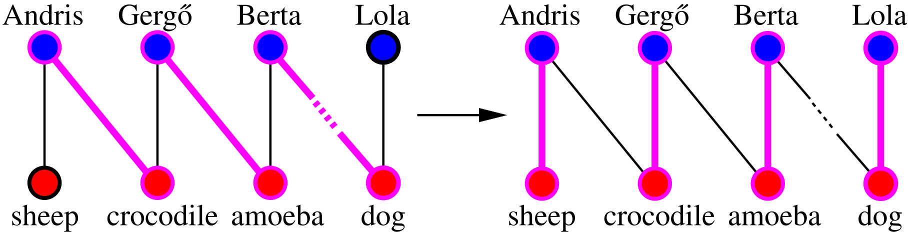

- "Pisti liked the spider on the table, so he picked it up."
- "Pisti swapped the spider for the camel. So the spider became free, which Marcsi liked, so she picked it up."
- "Andris swapped the crocodile for the sheep, then Gergő swapped the amoeba for the crocodile, etc. Finally, Lola picked up the dog."

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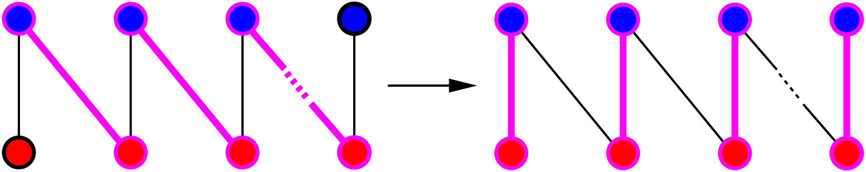

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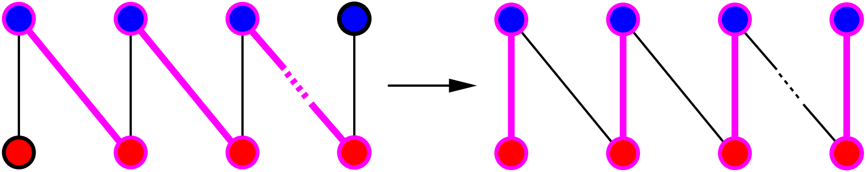

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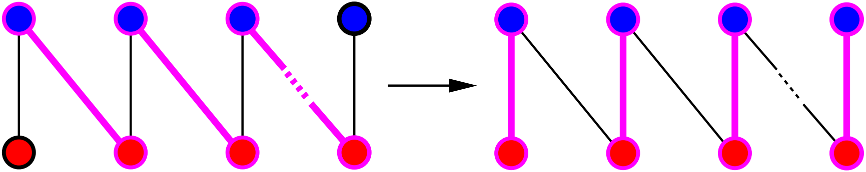

How could we build a method (algorithm) for finding a perfect matching from this notion?

- Start out from a random matching.
- Search for an augmenting path. If one is found, increase the matching by swapping along the path.
- Repeat this until no more augmenting path exists.

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- How? Use the algorithm!

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- It doesn't work, there is chaos and no perfect matching. (At least: hopefully. . .)

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- It doesn't work, there is chaos and no perfect matching. (At least: hopefully. . .)
- Why?
  - If we already have a (relatively big) matching, how do we search for an augmenting path?
  - How do we know that there is no more augmenting path?

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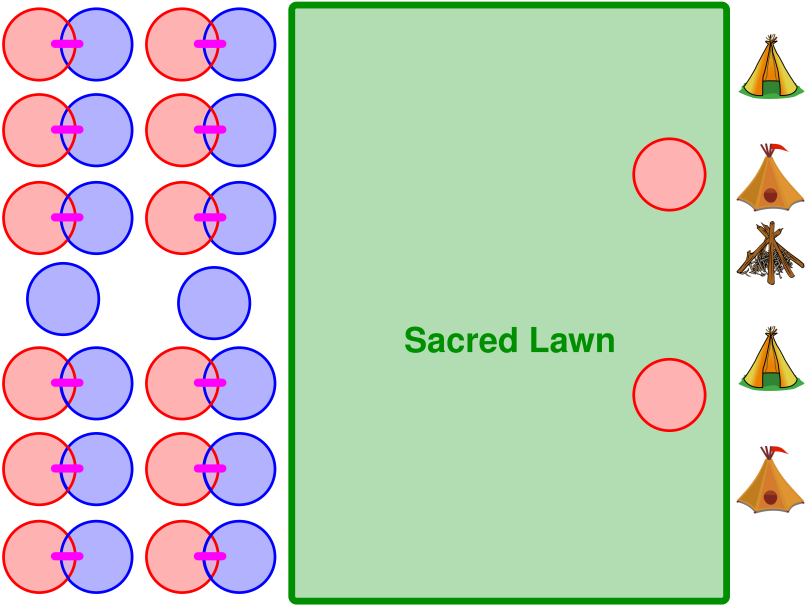

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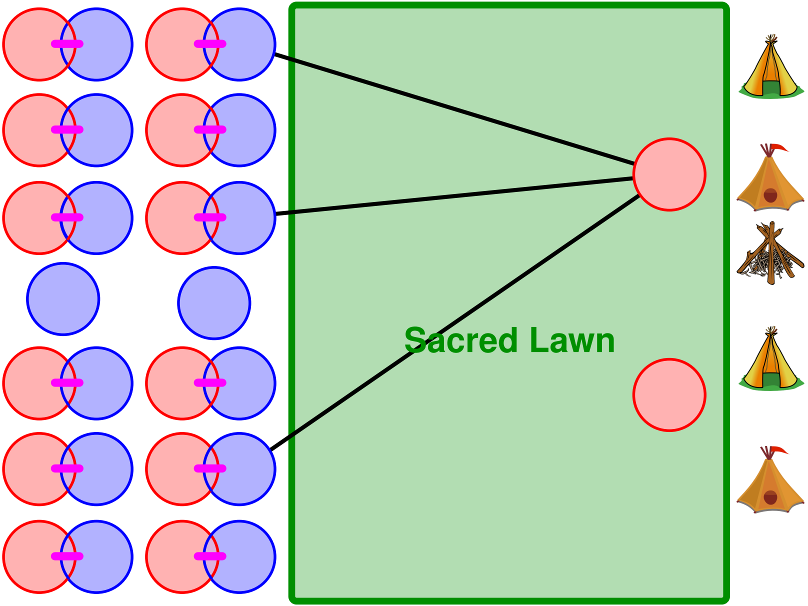

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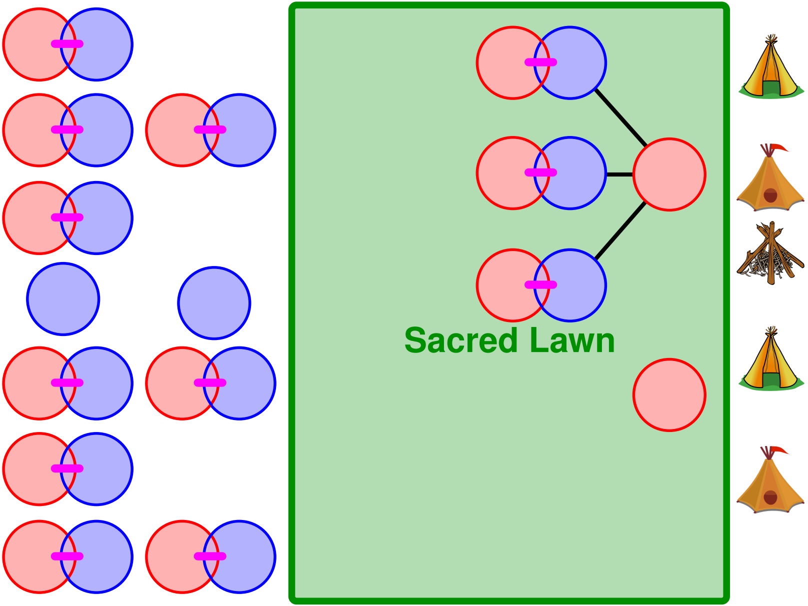

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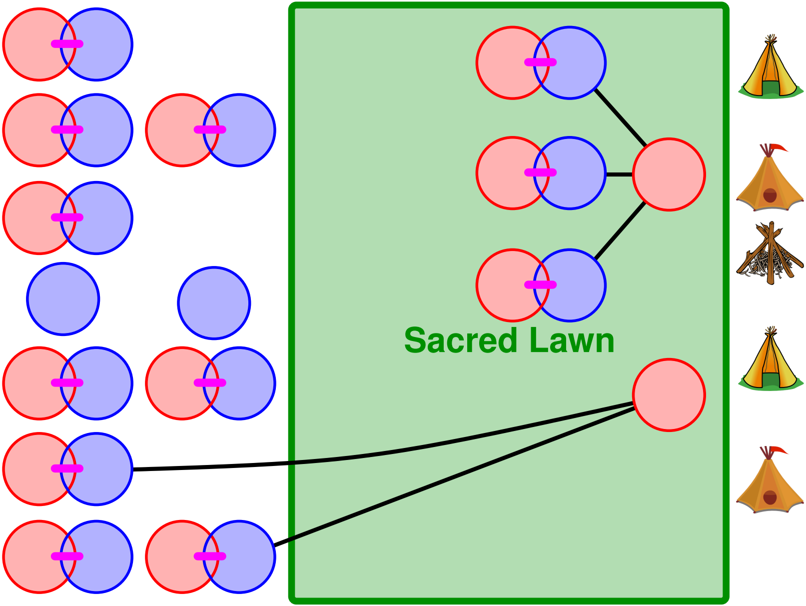

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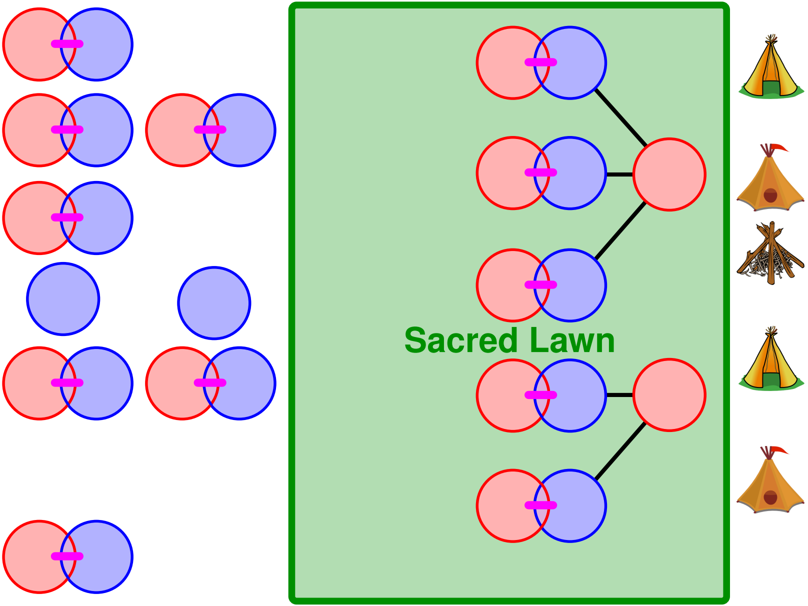

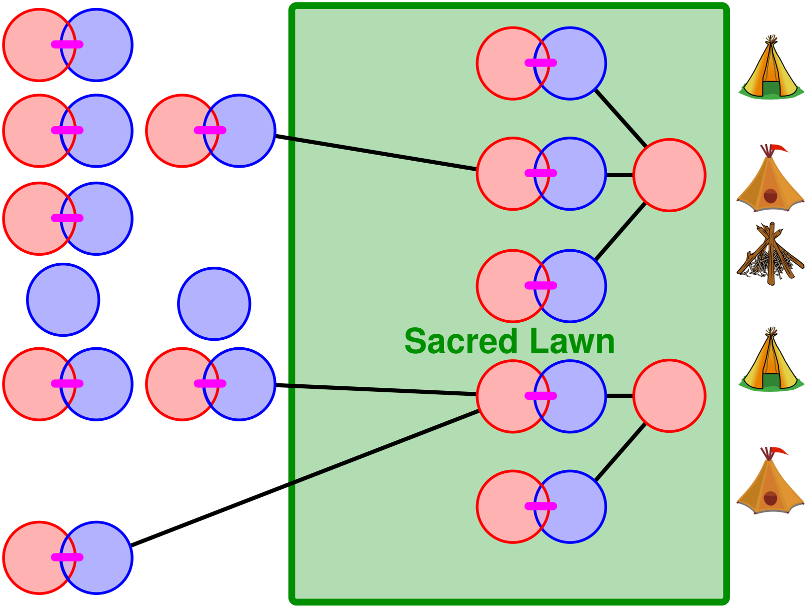
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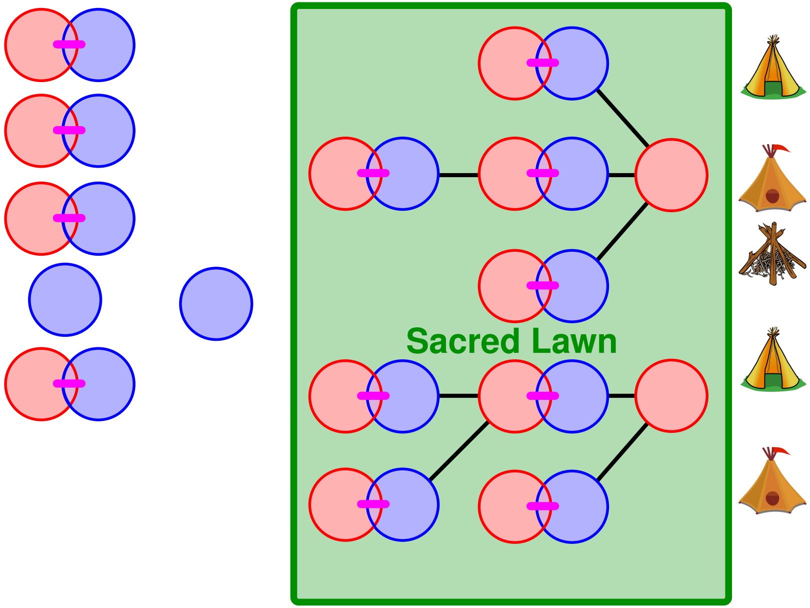

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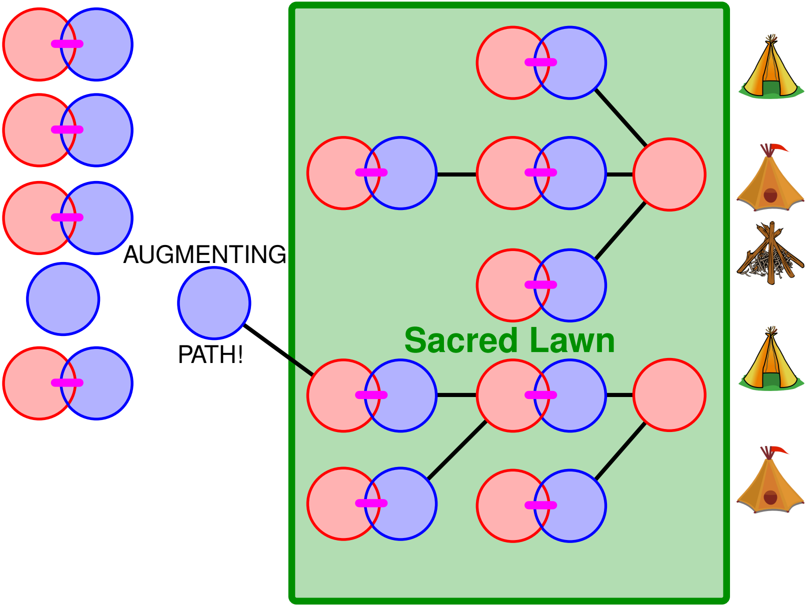

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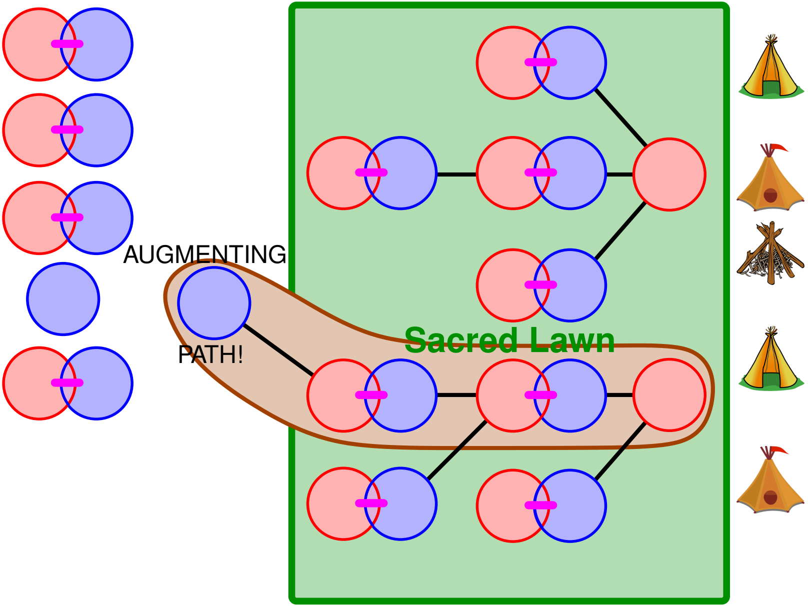

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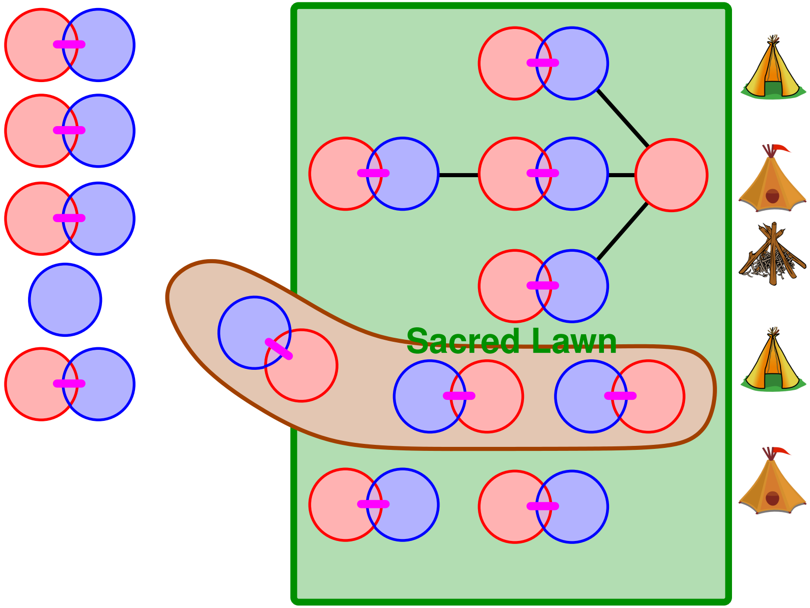

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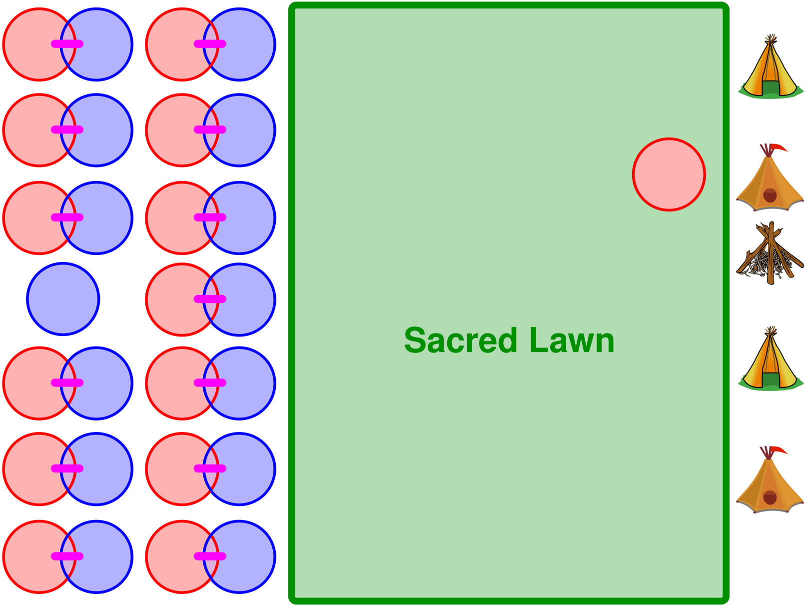

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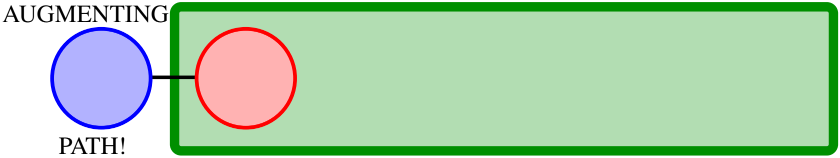

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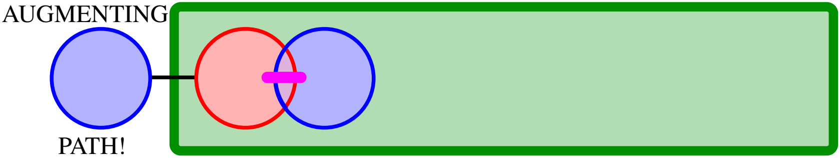

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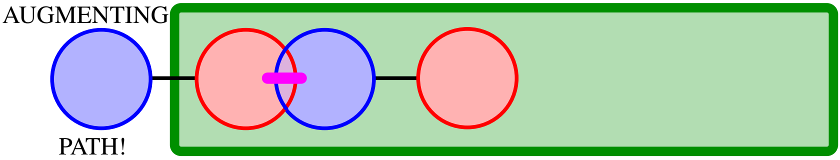

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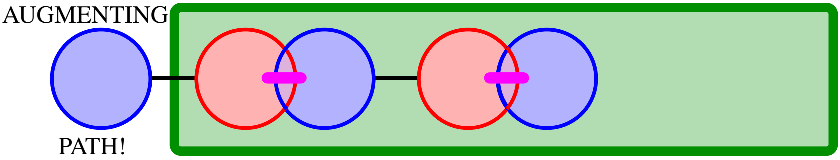

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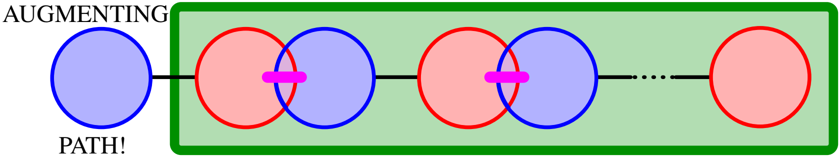

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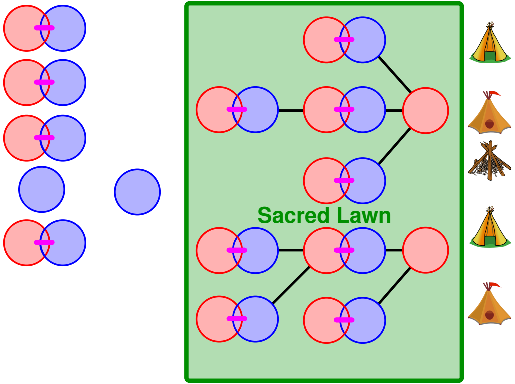
Is there really no perfect matching or is it just the algorithm that’s useless?

Hint:

- The answer to this question lies on the Sacred Lawn!



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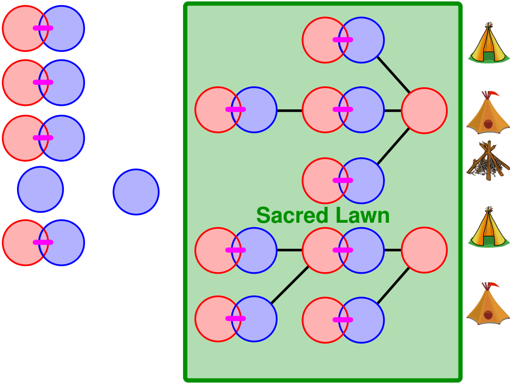

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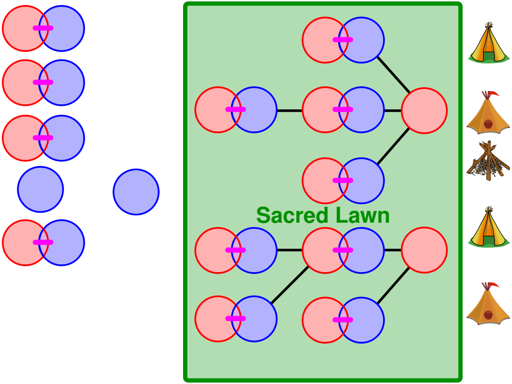
- Answer: girls on the lawn can only like boys on the lawn. What follows from this?



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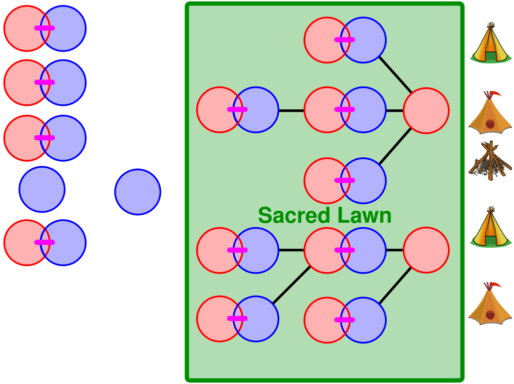
- What follows from the fact that the process is "frozen"?
- Answer: girls on the lawn can only like boys on the lawn. What follows from this?
- How does the number of girls and boys on the lawn compare? Is this always true in a similar situation?



4. Hall's Theorem

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What did we prove now? Formulate a theorem.

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Theorem (P. Hall, 1935)

In all such situations exactly one of the following is true:

- *There exists a perfect matching.*
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Either the Augmenting Path Algorithm finds a perfect matching, or if not then once it's over, girls on the Sacred Lawn only like boys on the Sacred Lawn, who are less in number.

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Ask interesting, relevant questions based on our discoveries.

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- Find the maximum number of pairs (between boys and girls). (Answer: the augmenting path algorithm finds one.) Proof: (not that easy) $\longrightarrow$ Problem.)
- Every (boy,girl) pair has a given profit value, we want a matching of maximum total profit.

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Ask interesting, relevant questions based on our discoveries.

- The number of boys and girls is not equal, we want a husband for every girl (but boys can remain single). (Answer: it's the same, both Hall's Theorem and its proof remain valid.)
- Find the maximum number of pairs (between boys and girls). (Answer: the augmenting path algorithm finds one.) Proof: (not that easy) $\longrightarrow$ Problem.)
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  NP-hard problem (that is, a *very* hard problem)

5. Problem Solving Session

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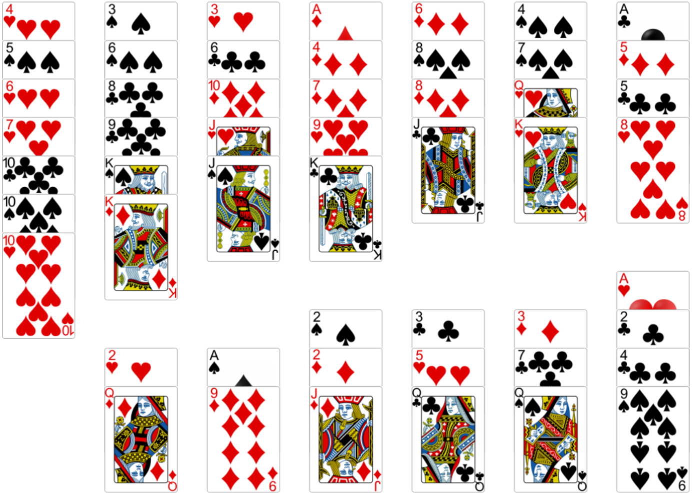
4. Is there an interesting special case in which this type of problem is always solvable? Find one. Prove that the problem is really solvable in these cases.

5. First Problem Set: Solitaire

1. A standard (52-card) deck of playing cards is dealt out into 13 smaller decks. Choose one card from each deck such that there is one of each rank among the selected cards (that is, one 2, one 3, one 4, etc., one Queen, one King and one Ace). (Suits of the cards are of no relevance.)



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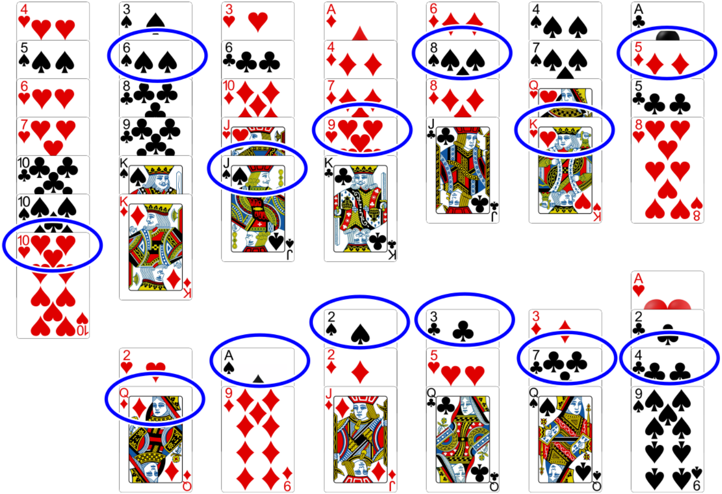

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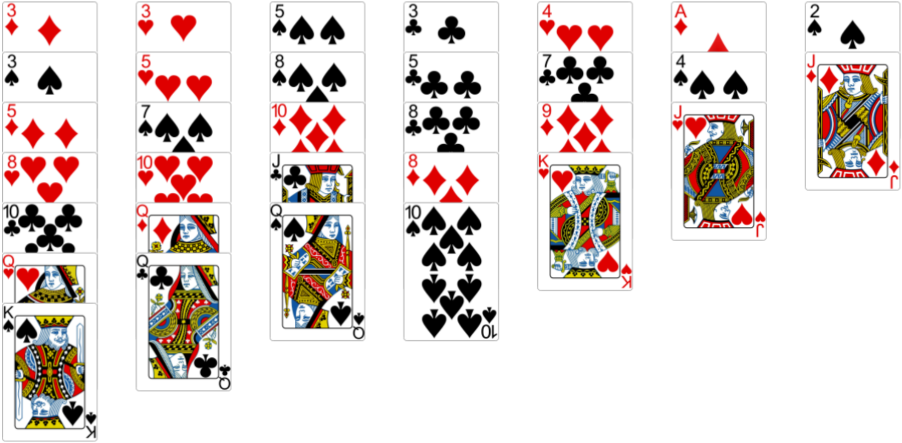

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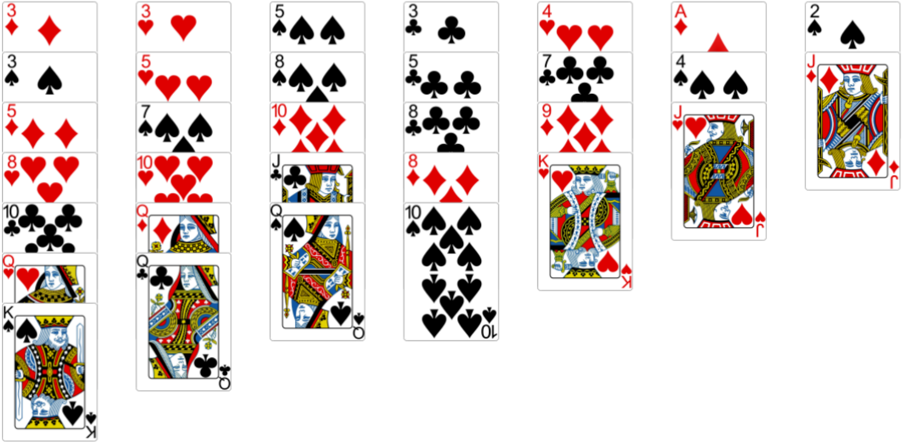

Solution: impossible, because each card of rank 3, 5, 8, 10 and Queen is in the first four decks.

3. Formulate a conjecture. Then try to prove it.

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Theorem

In all cases exactly one is true:

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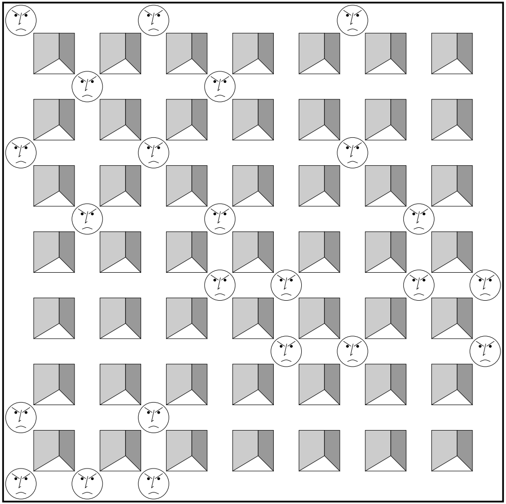
(When did we meet this phenomenon in a different form?
$\longrightarrow$ Plush Animal Game)

1. A small town consists of 8 North-South and 8 East-West streets. Security is very important for the inhabitants of the town so they built watchposts into some crossings – these are represented by angry faces. If a guard is deployed in a watchpost then both streets that cross at that watchpost can be kept under surveillance. Besides security, money is also valued by the inhabitants therefore they want to keep the whole town under surveillance by hiring not more than 8 guards and deploying them into suitably chosen watchposts.

Is that possible?



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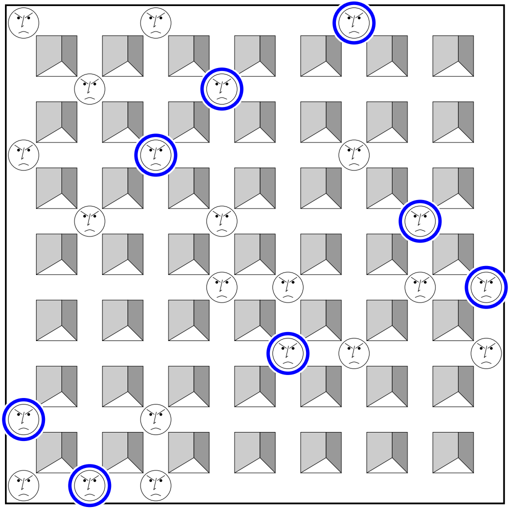
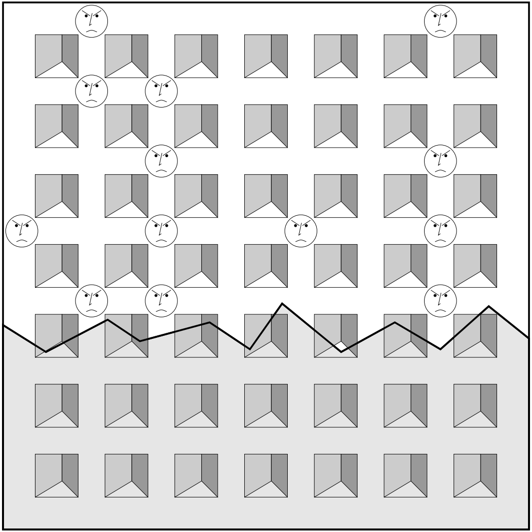


Is that possible?

5. Second Problem Set: Surveillance

2. This problem is like the previous one, however, the three southmost streets of the town were swept over by a tornado and all watchposts in that region were destroyed. Try to reconstruct the missing watchposts in such a way that the problem becomes solvable (with 8 guards).



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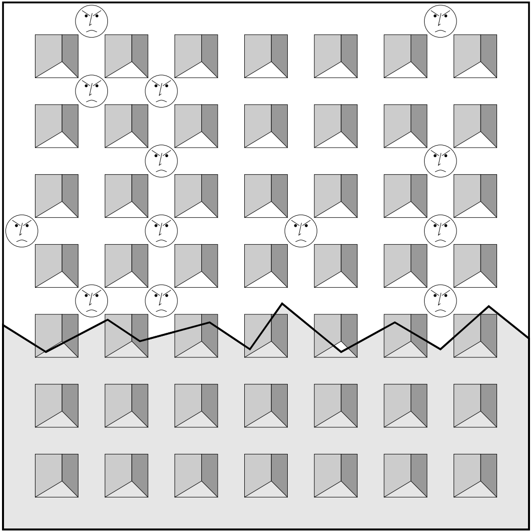

Solution: impossible because in the 1st, 2nd, 3rd and 5th East-West streets watchposts could only be deployed in the 2nd, 3rd or 7th North-South streets.

3. Formulate a conjecture. Then try to prove it.

Theorem

In all cases exactly one is true:

- *The Surveillance Problem is solvable.*
- *There exist k East-West streets such that all watchposts in these streets are positioned in less than k North-South streets.*

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4. Find an interesting special case in which this type of problem is always solvable.

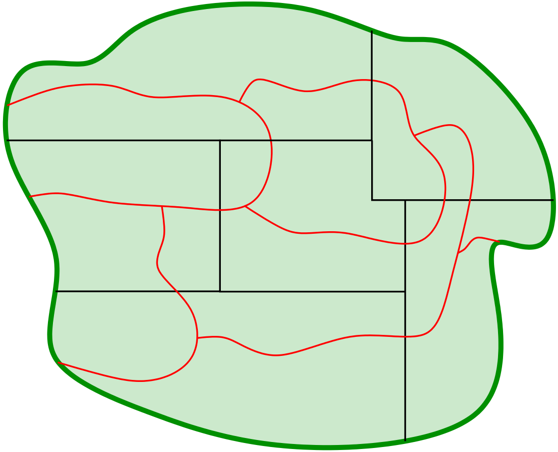
Theorem

If every (East-West and North-South) street contains the same number of watchposts then the Surveillance Problem is always solvable.

5. Third Problem Set: Settling on the Island

1. 6 tribes dwell on an island, they support themselves by cultivation and hunting. Due to frequent clashes between the tribes, the Ministry of Agriculture divided up the island into 6 parcels with the intention of giving one to each tribe for cultivation: these parcels are bounded by straight lines. Not knowing of the existing division, the Ministry of Hunting also parcelled out the island into 6 parts, these are bounded by curvy lines. Now every tribe gets one parcel for cultivation and another one for hunting. Obviously, the tribes want to distribute the parcels such that each tribe's two parcels have a common part where they can settle.

Is that possible?



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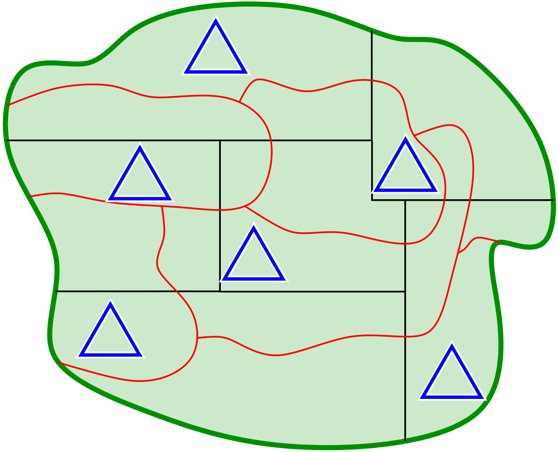

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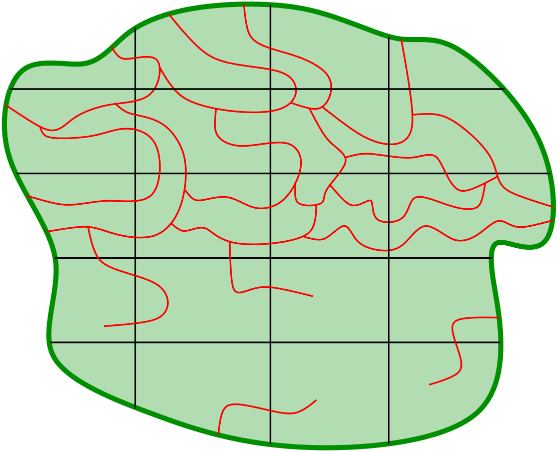
2. This problem is like the previous one with two differences. Firstly, now 20 tribes dwell on the island. Secondly, there have been cutbacks at the Ministry of Hunting, the map drawers were fired and thus their job remained unfinished. Try to finish their job in such a way that it will be possible to distribute the parcels to the satisfaction of all 20 tribes.



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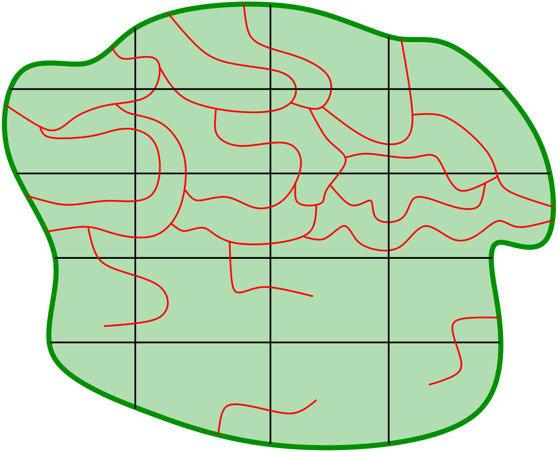
Solution: impossible, because the upper 3×4 cultivation parcels completely cover 13 hunting parcels.



3. Formulate a conjecture. Then try to prove it.

Theorem

In all cases exactly one is true:

- *The Settling on the Island Problem is solvable.*
- *There exits k hunting parcels that are completely covered by less than k cultivation parcels.*

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4. Find an interesting special case in which this type of problem is always solvable.

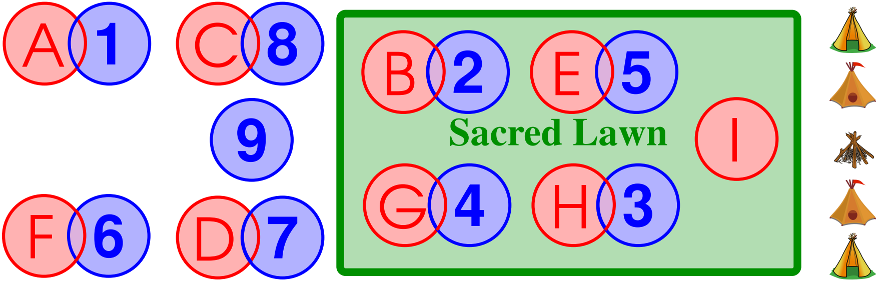
Theorem

If the area of all (hunting and cultivation) parcels are equal then the Settling on the Island Problem is always solvable.

Closing Remarks

Instead of party hats: an alternative way to present the Matching Game

- Instead of flesh-and-blood participants, boys and girls are represented by (blue and red, numbered and lettered) pawns



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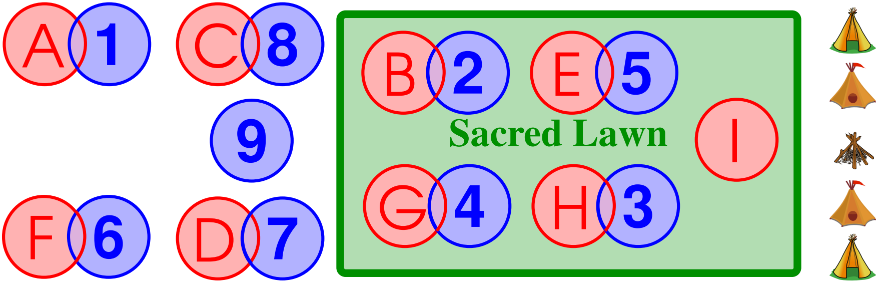
- Likings are given in a table, the Sacred Lawn is printed on paper.



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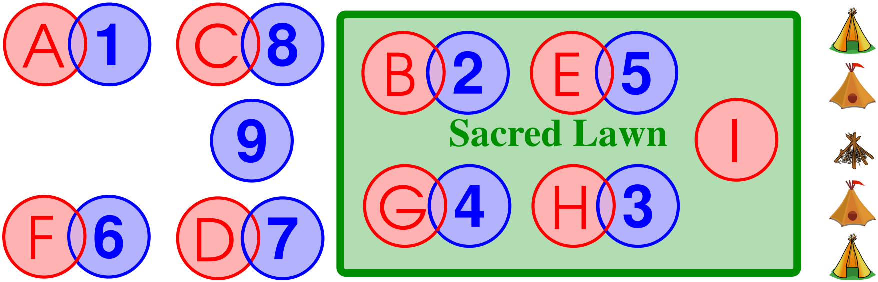
- Likings are given in a table, the Sacred Lawn is printed on paper.
- The algorithm is played in pairs: one player can only move boys, the other one girls.



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- According to our experience, the following works well: a high percentage of the edges is *critical*, that is, it is included in every perfect matching.

Behind the scenes: how was the graph of the Matching Game made?

		Girls										
		A	B	C	D	E	F	G	H	I	J	K
Boys	1	♥		♥	♥				♥			♥
	2	♥	♥	♥		♥		♥				
	3		♥	♥	♥		♥					
	4				♥	♥	♥	♥				
	5					♥		♥	♥		♥	
	6						♥		♥	♥		♥
	7							♥	♥	♥	♥	
	8								♥	♥		♥
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	5					❤		♥	♥		♥	
	6						❤		♥	♥		♥
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Extra Problems (for the apt and worthy)

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- Since every couple in every matching "consumes" at least one member of Z, no matching bigger than k can exist.

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25 little bugs live on a 5×5 *checkerboard, one on each square. At a certain moment, each bug sets off and moves to an adjacent square of the board. (By "adjacent" we mean that the squares share a common edge, we do not consider them to be adjacent if they only share a common vertex.) The bugs want to organize their (simultaneous) movement in such a way that finally each square will again be occupied by a single bug. Is this possible?*

After finding the answer for the above question, try to ask and answer further questions inspired by the problem of the 25 bugs.

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- The conjecture suggested by the checkerboard problem: if and only if no matter how *k* vertices are chosen from *G*, there exist at most *k* nodes among the remaining (non-chosen) vertices such that all their neigbours are among the *k* chosen ones.

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- The conjecture is true. Necessity is trivial, sufficiency can be proved using Hall's theorem (but it's non-trivial, it takes some work).

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In a school, each class and each teacher has at most k classes every week. Prove that a timetable with k time slots can be made.

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(That is: the edge-chromatic index of a bipartite graph is the maximum degree.)

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(That is: the edge-chromatic index of a bipartite graph is the maximum degree.)

Solution (sketch)

- We rely on: every regular (that is, every degree is the same) bipartite graph has a perfect matching. This follows from Hall's Theorem (and it also came up at the Solitaire and the Surveillance Problems).

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Extra Problems (for the apt and worthy)

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- If the graph is k-regular then we are done by applying this k times.
- If not then the graph can be augmented by extra vertices and edges to be k-regular. (For this, parallel edges have to be allowed, but that's not a problem.)

Extra Problems (for the apt and worthy)

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Let F and T be two spanning trees in the connected graph G with n vertices. Prove that the edges of F and T can be numbered from f_1 to f_{n-1} and t_1 to t_{n-1}, respectively, such that $(E(F) \setminus \{f_i\}) \cup \{t_i\}$ is also the edge set of a spanning tree in G for every $1 \leq i \leq n-1$.

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Solution (sketch)

Define a bipartite graph such that its two partition classes are $E(F)$ and $E(T)$ and f_i is adjacent to t_j if and only if $(E(F) \setminus \{f_i\}) \cup \{t_j\}$ is the edge set of a spanning tree. Apply Hall's Theorem in this graph to prove the existence of a perfect matching.

Extra Problems (for the apt and worthy)

Problem

Extend the notions of matching and augmenting path to arbitrary (non-bipartite) graphs and then answer the question: is it true that if there exists no augmenting path with respect to a matching then the matching is maximum?

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- Since $|M_2| > |M_1|$, one component of H is an augmenting path.
- Comment: although true, obtaining an efficient algorithm from this claim is much more complicated than in case of bipartite graphs (since searching for, or deciding the existence of an augmenting path is highly non-trivial). ($\longrightarrow$ *Edmonds' algorithm*)